

REPETITIVE RESOLUTIONS OVER CLASSICAL ORDERS AND FINITE DIMENSIONAL ALGEBRAS

K. R. GOODEARL AND B. HUISGEN-ZIMMERMANN

ABSTRACT. Repetitiveness in projective and injective resolutions and its influence on homological dimensions are studied. Some variations on the theme of repetitiveness are introduced, and it is shown that the corresponding invariants lead to very good – and quite accessible – upper bounds on various finitistic dimensions in terms of individual modules. These invariants are the ‘repetition index’ and the ‘syzygy type’ of a module M over an artinian ring Λ . The repetition index measures the degree of repetitiveness among non-projective direct summands of the syzygies of M , while the syzygy type of M measures the number of indecomposable modules among direct summands of the syzygies of M . It is proved that if T is a right Λ -module which contains an isomorphic copy of $\Lambda/J(\Lambda)$, then the left big finitistic dimension of Λ is bounded above by the repetition index of T , which in turn is bounded above by the syzygy type of T .

The finite dimensional K -algebras $\Lambda = \mathcal{O}/\pi\mathcal{O}$, where \mathcal{O} is a classical order over a discrete valuation ring D with uniformizing parameter π and residue class field K , are investigated. It is proved that, if $\text{gl.dim.}\mathcal{O} = d < \infty$, then the global repetition index of Λ is $d - 1$ and all finitely generated Λ -modules have finite syzygy type. Various examples illustrating the results are presented.

I. INTRODUCTION

It has long been known that a great deal of homological information can be gleaned from repetitions occurring in projective and injective resolutions. A fundamental – and ancient – example is the $(\mathbb{Z}/p^n\mathbb{Z})$ -module $\mathbb{Z}/p\mathbb{Z}$, whose successive syzygies alternate between $p\mathbb{Z}/p^n\mathbb{Z}$ and $\mathbb{Z}/p\mathbb{Z}$; from this phenomenon one immediately deduces that the module has infinite projective dimension. Numerous authors have systematized various kinds of repetitive homological behavior, e.g., with the notion of an ‘ultimately closed’ projective resolution which is due to Jans [13], or that of a resolution ‘with a strongly redundant image’ which was introduced by Colby and Fuller [4] (see also [6] and [7]). The consequences of various types of repetitive homology have also been studied by Igusa-Zacharia [12], Kirkman-Kuzmanovich-Small [19], and Wilson [24], to name a few other instances among many.

In the first part of this note, we introduce some variations on the theme of repetitiveness and show that the corresponding invariants lead to very good – and quite accessible – upper bounds on various finitistic dimensions in terms of individual modules. These invariants

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are the ‘repetition index’ and the ‘syzygy type’ of a module M over an artinian ring Λ . The repetition index, $\text{rep}(M)$, measures the degree of repetitiveness among non-projective direct summands of the syzygies of M ; to say that $\text{rep}(M) = n < \infty$, means tightening somewhat the Colby-Fuller concept of a ‘projective resolution with a strongly redundant image from n ’. The syzygy type of M measures the number of indecomposable modules among direct summands of the syzygies of M ; in case the module M is finitely generated, finiteness of its syzygy type is equivalent to the existence of an ultimately closed projective resolution in the sense of Jans. We prove that if T is a right Λ -module which contains an isomorphic copy of $\Lambda/\text{rad}(\Lambda)$, then the left big finitistic dimension of Λ is bounded above by $\text{rep}(T)$, which in turn is bounded above by the syzygy type of T . On one hand, this point of view leads to several methods for bounding finitistic dimensions; on the other, it frequently allows us to reduce the computation of a host of projective dimensions to the task of resolving a single module.

Our second goal here is to present a new class of algebras Λ for which the global repetition index (the supremum of the repetition indices of all Λ -modules) is finite. These are the finite dimensional K -algebras $\Lambda = \mathcal{O}/\pi\mathcal{O}$, where \mathcal{O} is a classical order over a discrete valuation ring D with uniformizing parameter π and residue class field K . The homological properties of \mathcal{O} are to a great extent determined by those of Λ , as was observed in [22], [15], and [8], while the latter algebra is substantially easier to handle. We are indebted to E. Kirkman and J. Kuzmanovich for bringing the crucial role played by these algebras to our attention. From our main result (Theorem 4.3) it follows that, if $\text{gl dim } \mathcal{O} = d < \infty$, then $\text{lgl rep } \Lambda = d - 1$ and all finitely generated Λ -modules have finite syzygy type. To illustrate the transport of information between \mathcal{O} and Λ , we present an example in which the left and right finitistic dimensions of \mathcal{O} differ, and one in which \mathcal{O} has infinite global dimension while Λ is Gorenstein; in particular, $\text{l fin dim } \mathcal{O} = \text{r fin dim } \mathcal{O}$ in this latter example.

Finally, we wish to advertize the algebras $\mathcal{O}/\pi\mathcal{O}$ within the representation theory community, as constituting a very interesting class of finite dimensional algebras deserving further study, both for their own sake and for their impact on the theory of classical orders.

Terminology. Given an artinian ring Λ , we denote by $\Lambda\text{-Mod}$ and $\Lambda\text{-mod}$ the category of all left Λ -modules and the full subcategory of finitely generated left Λ -modules, respectively. Moreover, given a class \mathcal{A} of left Λ -modules, the notation $\text{add } \mathcal{A}$ will stand for the full subcategory of $\Lambda\text{-Mod}$ having as objects all finite direct sums of copies of direct summands of objects from \mathcal{A} .

In addition to the global dimensions, we will consider the big and little finitistic dimensions, $\text{l Fin dim } \Lambda$ and $\text{l fin dim } \Lambda$; the former is the supremum of all the finite projective dimensions attained on objects in $\Lambda\text{-Mod}$, while the latter is the analogous supremum in $\Lambda\text{-mod}$.

Throughout, J will be the Jacobson radical of Λ .

We next present a few known results which will be used in the sequel.

Proposition 1.1. *If Λ is left noetherian, then $\text{l Fin dim } \Lambda \leq \text{i dim } \Lambda$.*

Proof. [3], Proposition 4.3. \square

Proposition 1.2. *If Λ is noetherian and $\text{idim } {}_{\Lambda}\Lambda$ and $\text{idim } \Lambda_{\Lambda}$ are both finite, then $\text{lFin dim } \Lambda = \text{rFin dim } \Lambda = \text{idim } {}_{\Lambda}\Lambda = \text{idim } \Lambda_{\Lambda}$.*

Proof. [25], Lemma A and [19], Proposition 2.1. \square

In case Λ is a finite dimensional algebra, there is an easier proof of the above result, which, moreover, yields some additional information:

Proposition 1.3. *Let Λ be a finite dimensional algebra over a field K , with Jacobson radical J . If $\text{idim } {}_{\Lambda}\Lambda$ and $\text{idim } \Lambda_{\Lambda}$ are both finite, then*

$$\begin{aligned} \text{lFin dim } \Lambda &= \text{rFin dim } \Lambda = \text{lfin dim } \Lambda = \text{rfin dim } \Lambda \\ &= \text{idim } {}_{\Lambda}\Lambda = \text{idim } \Lambda_{\Lambda} = \text{pdim } E({}_{\Lambda}(\Lambda/J)) = \text{pdim } E((\Lambda/J)_{\Lambda}). \end{aligned}$$

Proof. If $\text{idim } {}_{\Lambda}\Lambda = m$ and $\text{idim } \Lambda_{\Lambda} = n$, then $\text{lFin dim } \Lambda \leq m$ and $\text{rFin dim } \Lambda \leq n$ by Proposition 1. The duality $\text{Hom}_K(-, K)$ carries ${}_{\Lambda}\Lambda$ to the right minimal injective cogenerator $E((\Lambda/J)_{\Lambda})$, and so

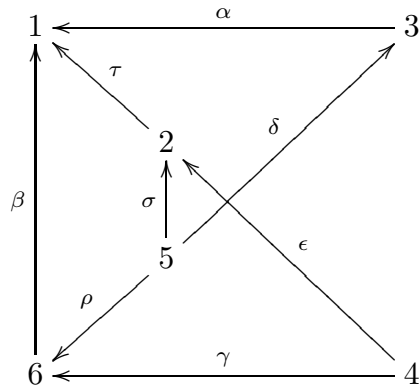
$$m = \text{pdim } E((\Lambda/J)_{\Lambda}) \leq \text{rfin dim } \Lambda \leq \text{rFin dim } \Lambda \leq n$$

because $E((\Lambda/J)_{\Lambda})$ is finitely generated. Similarly,

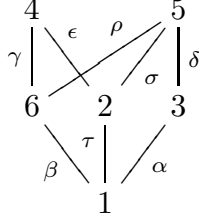
$$n = \text{pdim } E({}_{\Lambda}(\Lambda/J)) \leq \text{lfin dim } \Lambda \leq \text{lFin dim } \Lambda \leq m. \quad \square$$

Graphing conventions. For the convenience of the reader, we include a sketch of our graphing conventions for modules over path algebras modulo relations; these may differ from standard ones in that they take the J -layering of the modules into account. Suppose that $\Lambda = K\Gamma/I$ is a path algebra modulo relations over a field K , where Γ is a quiver and I an admissible ideal in $K\Gamma$. We will label the vertices of Γ by integers, and denote the primitive idempotent in Λ corresponding to a vertex i by e_i ; further, we will often write S_i for the simple left Λ -module $\Lambda e_i / J e_i$.

For example, suppose that Γ is the quiver



That a left Λ -module M has graph



is to communicate the following information: First,

$$M/JM \cong S_4 \oplus S_5; \quad JM/J^2M \cong S_6 \oplus S_2 \oplus S_3; \quad J^2M \cong S_1; \quad J^3M = 0.$$

Second, there exist elements $x = e_4x$ and $y = e_5y$ in $M \setminus JM$ such that $\gamma x, \rho y \in JM \setminus J^2M$ and these last two elements differ only by a nonzero scalar factor modulo J^2M ; analogously, ϵx and σy belong to $JM \setminus J^2M$ and differ only by a nonzero scalar modulo J^2M . Finally, $\beta\gamma x, \tau\epsilon x, \alpha\delta y$ are to differ only by nonzero scalar factors, that is, $\Lambda\beta\gamma x = \Lambda\tau\epsilon x = \Lambda\alpha\delta y = J^2M$ in this case. (It follows from the information in the previous ‘layer’ that $\beta\rho y$ and $\tau\sigma y$ are automatically included in this list.)

Note that, in general, a module M will have more than one graph, since the latter may depend on the choice of the ‘reference elements’ in $M \setminus JM$ (the elements x and y in the example above). In case there is just one arrow from a vertex i to a vertex j , we will often omit naming this arrow. For additional information, we refer to [11].

Finally, observe that right Λ -modules correspond to left modules over a quotient of the path algebra of the ‘dual’ of the quiver Γ , that is, of the quiver with the same vertices as Γ but reversed arrows. Thus in the graph of a right Λ -module, an edge leading from a vertex j down to a vertex i corresponds to an arrow $i \rightarrow j$, rather than to an arrow $j \rightarrow i$.

2. SYZGY TYPE AND REPETITION INDEX

Throughout this section, let Λ be an artinian ring. There are analogs of many of the results of this section for finitely generated modules over a noetherian semiperfect ring – see Remark 2.10.

Definitions 2.1. Let M and A be left Λ -modules.

(1) Let n be a nonnegative integer. We shall say that the minimal projective resolution of M is *repetitive at degree n* if there exists a decomposition $\Omega^n(M) = P \oplus \bigoplus_{i \in I} A_i$ such that P is projective and each A_i occurs as a direct summand of infinitely many $\Omega^j(M)$. In case $\Omega^n(M)$ is finitely generated, this is the same as to say that each non-projective indecomposable direct summand of $\Omega^n(M)$ occurs as a direct summand of infinitely many $\Omega^j(M)$.

(2) The *repetition index* of M , denoted $\text{rep}(M)$, is the least nonnegative integer k such that the minimal projective resolution of M is repetitive at degree k (if such a k exists) or

∞ (otherwise). The corresponding globalized indices are

$$\begin{aligned} \text{lg l rep}(\Lambda) &= \sup\{\text{rep}(M) \mid M \in \Lambda\text{-Mod}\} \\ \text{l Fin rep}(\Lambda) &= \sup\{\text{rep}(M) \mid M \in \Lambda\text{-Mod and } \text{rep}(M) < \infty\} \\ \text{l fin rep}(\Lambda) &= \sup\{\text{rep}(M) \mid M \in \Lambda\text{-mod and } \text{rep}(M) < \infty\}, \end{aligned}$$

which we shall call the *left global repetition index*, the *left big finitistic repetition index*, and the *left little finitistic repetition index* of Λ , respectively.

(3) Next, we set $\sigma_M(A) = -1$ if A is not isomorphic to a direct summand of any syzygy $\Omega^j(M)$, $j \geq 0$, and

$$\sigma_M(A) = \sup\{j \geq 0 \mid A \cong \text{a direct summand of } \Omega^j(M)\}$$

otherwise. We call $\sigma_M(A)$ the *(homological) contingency of A to M* .

Clearly $\text{rep}(M)$ equals the minimum of those $k \geq 0$ such that $\Omega^k(M)$ is a direct sum of a projective module and modules with infinite contingency to M .

(4) The *syzygy category* of M is the full subcategory $\text{add}(\{\Omega^j(M) \mid j \geq 0\})$ of $\Lambda\text{-Mod}$. If this category is a Krull-Schmidt category of finite representation type, then we say that M has *finite syzygy type*, and we define the *syzygy type* of M to be the number of isomorphism types of nonzero indecomposable objects in the syzygy category. Otherwise, we say that the syzygy type of M is ∞ .

(5) For any nonnegative integer τ , let $\Omega^\tau(\Lambda\text{-Mod})$ denote the full subcategory

$$\text{add}(\{\Omega^j(M) \mid j \geq \tau; M \in \Lambda\text{-Mod}\})$$

of $\Lambda\text{-Mod}$. As in (4), if $\Omega^\tau(\Lambda\text{-Mod})$ is a Krull-Schmidt category of finite representation type, then we say that $\Lambda\text{-Mod}$ has *finite syzygy type from degree τ* , and we define the *syzygy type of $\Lambda\text{-Mod}$ from degree τ* to be the number of isomorphism types of nonzero indecomposable objects in the category $\Omega^\tau(\Lambda\text{-Mod})$.

Comments 2.2. (a) The earliest consideration of these conditions goes back to Jans [13], who said that a Λ -module M has an *ultimately closed projective resolution* in case some syzygy $\Omega^n(M)$, $n \geq 1$, has a decomposition $\Omega^n(M) = \bigoplus_{i \in I} A_i$ such that each A_i already occurred as a direct summand of a previous syzygy $\Omega^{j(i)}(M)$ with $j(i) < n$. It is easily seen that, for a finitely generated module M , this is the same as to say that M has finite syzygy type. Subsequently, Colby-Fuller [4] and Fuller-Wang [7] relaxed this condition slightly as follows: M is said to have a projective resolution with a *strongly redundant image from an integer $n \geq 1$* in case the n -th syzygy has a decomposition $\Omega^n(M) = \bigoplus_{i \in I} A_i$ such that each A_i is a direct summand of a later syzygy $\Omega^{j(i)}(M)$ with $j(i) > n$. In case $\Omega^n(M)$ has a decomposition into modules with local endomorphism rings, this latter condition clearly implies $\text{rep}(M) \leq n$. Note that strict inequality may occur, because we exclude projective direct summands of the $\Omega^j(M)$'s from consideration in $\text{rep}(M)$. We point to [4] for an example of a projective resolution with a strongly redundant image which fails to be ultimately closed.

(b) Observe that the restriction to non-projective direct summands in our definition of ‘repetition index’ allows us to compute the repetition index of a finitely generated module M by looking at any projective resolution, not just a minimal one.

(c) If $\Lambda\text{-Mod}$ has finite syzygy type from some degree, then each left Λ -module has finite repetition index. In fact, $\text{lgl rep}(\Lambda) < \infty$ in that case – see Lemma 2.4(b).

(d) Note that $\text{p dim}(M) = \text{rep}(M)$ for any module M with finite projective dimension. Hence, we have the inequalities $\text{l fin dim } \Lambda \leq \text{l fin rep } \Lambda \leq \text{lgl rep } \Lambda$ and $\text{l Fin dim } \Lambda \leq \text{l Fin rep } \Lambda \leq \text{lgl rep } \Lambda$. Strict inequalities may occur in all places; for an instance where even $\text{l Fin dim } \Lambda < \text{l fin rep } \Lambda$, see Example 2.3(d).

Examples 2.3. (a) The module category $\Lambda\text{-Mod}$ of any finite dimensional monomial relation algebra $\Lambda = K\Gamma/I$ has finite syzygy type from degree 2, by [9], Theorem A; in fact, the syzygy type of $\Lambda\text{-Mod}$ from degree 2 is bounded above by the number of paths in $K\Gamma$ whose residue classes are nontrivial in Λ .

(b) If $\text{gl dim } \Lambda = m < \infty$, then Λ has finite syzygy type from degree m .

(c) If $\Lambda = K[X, Y]/(X^2, Y^2)$ for a field K , then Λ has infinite repetition index and, a fortiori, infinite syzygy type; in fact, $\text{rep}(\Lambda/J) = \infty$. On the other hand, $\text{Fin dim } \Lambda = \text{rep}(E(\Lambda/J)) = 0$.

(d) Let $\Lambda = K\Gamma/I$, where Γ is the quiver

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \begin{array}{c} \curvearrowright \\ \gamma \end{array}$$

and I is generated by all paths of length 3. It is easy to see that $\text{l fin dim } \Lambda = \text{l Fin dim } \Lambda = 0$ in this case (see [2], Theorem 6.3 and [10], Corollary 8). On the other hand, $\text{l fin rep}(\Lambda) \geq$

$\text{rep}(S_1) = 2$, since $\Omega^1(S_1)$ has graph $\begin{array}{c} 2 \\ | \\ 3 \end{array}$, while $\Omega^2(S_1) \cong \Omega^{2n}(S_1) \cong S_3$ for $n \geq 1$, and

$\Omega^3(S_1) \cong \Omega^{2n+1}(S_1)$ has graph $\begin{array}{c} 3 \\ | \\ 3 \end{array}$ for $n \geq 1$. Thus indeed, $\text{l Fin dim } \Lambda < \text{lgl rep } \Lambda$.

Finally, observe that the syzygy category of S_1 is the additively closed category generated by the indecomposable left Λ -modules $\Omega^i(S_1)$, $i = 0, 1, 2, 3$, in this example. \square

The principal connections among syzygy type, repetition index, and finitistic dimensions are as follows. Note that, in essence, most of the arguments are familiar – see, e.g., [12], [7]. In fact, for $T = \Lambda/J$, where Λ is an artin algebra, the first statement of Theorem 2.6 is implicit in [12].

Lemma 2.4. (a) *The repetition index of any Λ -module M is bounded above by the syzygy type of M .*

(b) *If $\Lambda\text{-Mod}$ has syzygy type s from degree τ , then $\text{lgl rep}(\Lambda) \leq \tau + s$.*

Proof. (a) Assume that the syzygy type of M is $s < \infty$. If $s = 0$, then $M = 0$ and hence $\text{rep}(M) = 0$.

Now assume that $s > 0$, and let $B = B_s$ be an indecomposable direct summand of $\Omega^s(M)$. Since M has finite syzygy type, $\Omega^{s-1}(M)$ is a direct sum of indecomposable modules I_k , whence $\Omega^s(M) \cong \bigoplus_k \Omega^1(I_k)$, and so B_s is isomorphic to a direct summand of some $\Omega^1(I_k)$. (Here we are using the assumption that the syzygy category of M is a Krull-Schmidt category.) Label this I_k as B_{s-1} . Continuing by induction, we obtain indecomposable modules B_s, B_{s-1}, \dots, B_0 such that each B_j is a direct summand of $\Omega^j(M)$ and B_j is isomorphic to a direct summand of $\Omega^1(B_{j-1})$ for $j > 0$.

By definition of s , there exist indices p, q with $0 \leq p < q \leq s$ such that $B_p \cong B_q$. Thus B_p is isomorphic to a direct summand of $\Omega^{q-p}(B_p)$, and hence isomorphic to a direct summand of $\Omega^{m(q-p)}(B_p)$ for all $m \geq 0$. Since B is isomorphic to a direct summand of $\Omega^{s-p}(B_p)$ and B_p is isomorphic to a direct summand of $\Omega^p(M)$, it follows that B is isomorphic to a direct summand of $\Omega^{s+m(q-p)}(M)$ for all $m \geq 0$. Therefore $\text{rep}(M) \leq s$, because B was an arbitrary indecomposable direct summand of $\Omega^s(M)$.

(b) It follows from our assumptions that for any $M \in \Lambda\text{-Mod}$, the module $\Omega^\tau(M)$ has syzygy type at most s . Thus $\text{rep}(\Omega^\tau(M)) \leq s$ by part (a), and therefore $\text{rep}(M) \leq \tau + s$. \square

Observation 2.5. *Whenever T is a right Λ -module such that Λ/J , viewed as a right Λ -module, embeds into T ,*

$$\text{lFin dim } \Lambda \leq \text{rep}(T).$$

Proof. Assume that $\text{rep}(T) = r < \infty$, and write $\Omega^r(T) = P \oplus \bigoplus_{i \in I} A_i$ where P is projective and $\sigma_T(A_i) = \infty$ for all i . Consider a left Λ -module M with $\text{p dim } M = d < \infty$, and note that $\text{Tor}_d^\Lambda(\Lambda/J, M) \neq 0$. By assumption, there exists a short exact sequence of right Λ -modules $0 \rightarrow \Lambda/J \rightarrow T \rightarrow C \rightarrow 0$. Since $\text{Tor}_{d+1}^\Lambda(C, M) = 0$, we must have $\text{Tor}_d^\Lambda(T, M) \neq 0$.

If $d > r$, then $\text{Tor}_{d-r}^\Lambda(\Omega^r(T), M) \neq 0$, and so $\text{Tor}_{d-r}^\Lambda(A_i, M) \neq 0$ for some i . Since $\sigma_T(A_i) = \infty$, it follows that $\text{Tor}_{d-r}^\Lambda(\Omega^j(T), M) \neq 0$ for infinitely many j , and consequently $\text{Tor}_{d-r+j}^\Lambda(T, M) \neq 0$ for infinitely many j . However, this is impossible for $j > r$. Therefore $d \leq r$.

Since M was an arbitrary left Λ -module of finite projective dimension, this proves that $\text{lFin dim } \Lambda \leq r$. \square

Theorem 2.6. *Suppose there exists a right Λ -module T of finite syzygy type s such that Λ/J , viewed as a right Λ -module, embeds into T . Moreover, let A_1, \dots, A_s be representatives for the isomorphism types of the indecomposable objects in the syzygy category of T . Then*

(a) $\text{lFin dim } \Lambda \leq \text{rep}(T) \leq s$.

(b) *If M is a left Λ -module of finite projective dimension and*

$$\mu = \max\{\sigma_T(A_i) \mid 1 \leq i \leq s \text{ and } \text{Tor}_1^\Lambda(A_i, M) \neq 0\},$$

then $\text{p dim}(M) = \mu + 1$. (Let $\mu = -1$ if the above set is empty.)

Proof. (a) follows from Lemma 2.4 and Observation 2.5.

(b) Set $d = \text{p dim } M < \infty$, and note that $\text{Tor}_d^\Lambda(T, M) \neq 0$, as in the previous proof.

Consider one of the A_i for which $\text{Tor}_1^\Lambda(A_i, M) \neq 0$, and choose an index j such that A_i is isomorphic to a direct summand of $\Omega^j(T)$. Then $\text{Tor}_{j+1}^\Lambda(T, M) \cong \text{Tor}_1^\Lambda(\Omega^j(T), M) \neq 0$, and so $j + 1 \leq d$. Thus $\sigma_T(A_i) + 1 \leq d$ for all A_i such that $\text{Tor}_1^\Lambda(A_i, M) \neq 0$, that is, $\mu + 1 \leq d$.

It remains to prove the reverse inequality. If $d = 0$, then $\text{Tor}_1^\Lambda(-, M) \equiv 0$, whence $\mu = -1$ and $d = \mu + 1$. If $d > 0$, then $\text{Tor}_1^\Lambda(\Omega^{d-1}(T), M) \cong \text{Tor}_d^\Lambda(T, M) \neq 0$, and so $\text{Tor}_1^\Lambda(A_i, M) \neq 0$ for some A_i which is isomorphic to a direct summand of $\Omega^{d-1}(T)$. In this case, $d - 1 \leq \sigma_T(A_i) \leq \mu$, and therefore $d \leq \mu + 1$. \square

Corollary 2.7. *Let T and A_1, \dots, A_s be right Λ -modules as in the hypotheses of Theorem 2.6. Then*

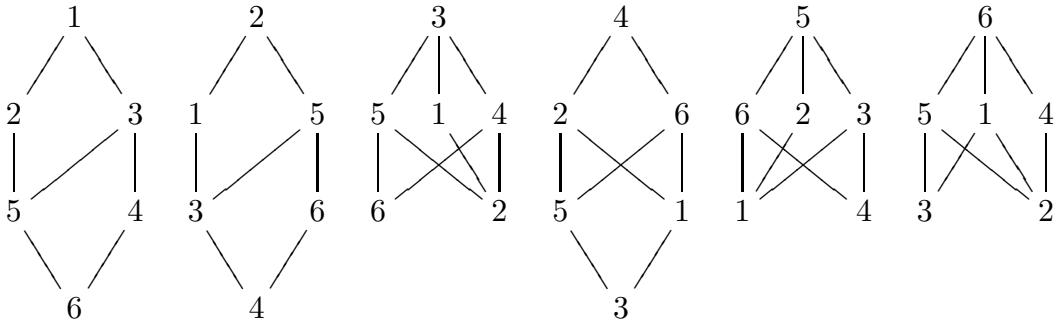
$$\text{lFindim } \Lambda \leq 1 + \max_{1 \leq i \leq s} \{\sigma_T(A_i) \mid A_i \text{ non-projective and } \sigma_T(A_i) < \infty\}. \quad \square$$

Of course, the module T in (2.5–2.7) may be chosen to be either the right module Λ/J or the minimal injective cogenerator E for the category of right Λ -modules. In [7], certain related bounds on the finitistic dimensions of noetherian rings are obtained. For the case where the base ring is an artin algebra, our Observation 2.5 provides a mild tightening of these bounds: For Theorem 3 of [7], take $T = E$; for Theorem 9, take $T = \Lambda/J$.

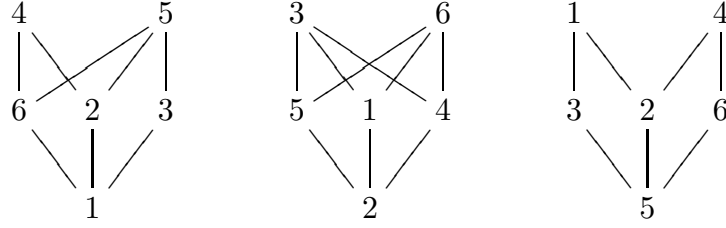
For reasons that are not transparent as yet, the choice $T = E$ appears to systematically yield better bounds on $\text{lFindim } \Lambda$. In particular, while it is easy to construct algebras with $\text{rep}(\Lambda/J) = \infty$ (see Example 2.3(c) above), examples where $\text{rep}(E) = \infty$ are not immediate.

The following example will illustrate the methods just introduced; further, it will be relevant to Example 4.6.

Example 2.8. Let $\Lambda = K\Gamma/I$ be a binomial relation algebra (i.e., I can be generated by paths in $K\Gamma$ and by differences $p - kq$ with $k \in K$, where p and q are paths) with six vertices such that the indecomposable projective right Λ -modules P_1, \dots, P_6 have the following graphs:



Then the indecomposable injective right Λ -modules E_i are $E_3 = P_4$, $E_4 = P_2$, $E_6 = P_1$, while E_1 , E_2 , E_5 have graphs

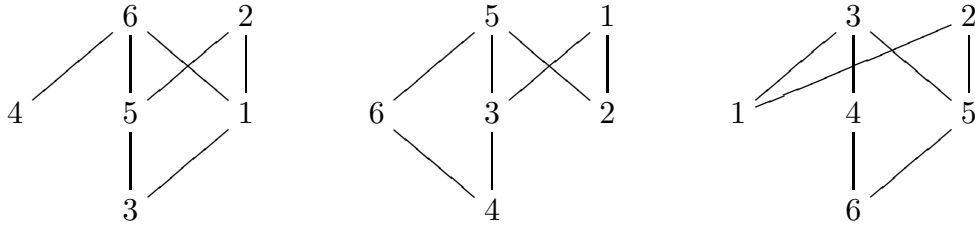


respectively. Set $E = E_1 \oplus \cdots \oplus E_6$. Clearly $\text{rep } E_3 = \text{rep } E_4 = \text{rep } E_6 = 0$. To compute $\text{rep } E$, we determine the minimal projective resolutions of E_1, E_2, E_5 :

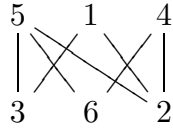
$\Omega^1(E_1)$

$\Omega^2(E_1)$

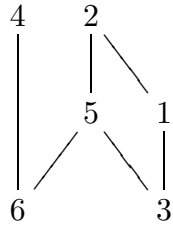
$\Omega^3(E_1)$



and $\Omega^4(E_1) \cong \Omega^2(E_1)$. Moreover, $\Omega^1(E_2)$ has graph



and $\Omega^2(E_2) \cong \Omega^1(E_1) \oplus \Omega^3(E_1)$. Finally, $\Omega^1(E_5)$ has graph



and $\Omega^2(E_5) \cong \Omega^1(E_1)$. It follows that $\text{rep } E = \text{rep}(E_1 \oplus E_2 \oplus E_5) = 3$, and hence Observation 2.5 yields $\text{l Fin dim } \Lambda \leq 3$. We actually obtain equality, $\text{l Fin dim } \Lambda = \text{l fin dim } \Lambda = 3$, since the indecomposable injective left Λ -module $E(\Lambda e_3 / J e_3)$ has projective dimension 3. Note that, on the other hand, $\text{r Fin dim } \Lambda = 0$, since the left socle of Λ contains a copy of Λ/J (see [2], Theorem 6.3 and [10], Corollary 8). \square

Often the following variant of Theorem 2.6 provides better bounds:

Corollary 2.9. *Suppose there exists a right Λ -module T of finite syzygy type such that Λ/J , viewed as a right Λ -module, embeds into T . For $m \geq 0$, let s_m be the syzygy type of $\Omega^m(T)$. Then $\text{lfin dim } \Lambda \leq s_m + m$ for all m .*

Proof. By Lemma 2.4, $\text{rep}(\Omega^m(T)) \leq s_m$, and so $\text{rep}(T) \leq s_m + m$. The corollary thus follows immediately from part (a) of Theorem 2.6. \square

Remark 2.10. Since the definitions and proofs in this section rely mainly on the existence and behavior of syzygies, the results carry over to the case when Λ is only assumed to be a noetherian semiperfect ring, provided we restrict attention to finitely generated modules. (Recall that semiperfect rings are precisely the rings over which every finitely generated module has a projective cover.) In particular, the argument of Lemma 2.4(a) shows that the repetition index of any finitely generated Λ -module is bounded above by its syzygy type, and the argument of Observation 2.5 shows that if T is a finitely generated right Λ -module containing an isomorphic copy of $(\Lambda/J)_{\Lambda/J}$, then

$$\text{lfin dim } \Lambda \leq \text{rep}(T).$$

Similarly, there are analogs of Theorem 2.6(b) (giving a formula for the projective dimensions of finitely generated Λ -modules with finite projective dimension) and of Corollaries 2.7, 2.9 (giving additional upper bounds for $\text{lfin dim } \Lambda$). We leave the details to the interested reader.

3. COMPUTATION OF INDIVIDUAL PROJECTIVE DIMENSIONS IN THE CASE OF FINITE SYZYGY TYPE

We now specialize to the case where Λ is a finite dimensional algebra over a field K . Suppose there exists a finitely generated right Λ -module T of finite syzygy type s such that $(\Lambda/J)_{\Lambda}$ embeds into T , and let A_1, \dots, A_s be representatives for the isomorphism types of the indecomposable objects in the syzygy category of T .

Guiding idea: If one knows the A_i and their first syzygies, the computation of projective dimensions of left Λ -modules M can be reduced to the calculation of the K -dimensions of the tensor products $A_i \otimes_{\Lambda} M$.

Indeed, by the choice of the A_i , there exists an $s \times s$ matrix $\mathbf{B} = (b_{ij})$ of nonnegative integers such that

$$\Omega^1(A_i) \cong \bigoplus_{j=1}^s A_j^{b_{ij}}$$

for all i . Moreover, let e_1, \dots, e_n be a full set of orthogonal primitive idempotents in Λ , and let P_i denote the projective cover of A_i . Write

$$P_i \cong \bigoplus_{l=1}^n (e_l \Lambda)^{p_{il}}$$

with $p_{il} \geq 0$ for $1 \leq i \leq s$.

Observe that since the entries of \mathbf{B} are nonnegative integers, the product of \mathbf{B} with any column vector of cardinal numbers is defined. (We use the convention that $0 \cdot \alpha = 0$.)

Proposition 3.1. *Let M be a left Λ -module, set $\tau_i = \dim_K \operatorname{Tor}_1^\Lambda(A_i, M)$ for $i = 1, \dots, s$, and set $\tau = \tau(M) = (\tau_1, \tau_2, \dots, \tau_s)^{\text{tr}}$.*

(a) *If M has finite projective dimension, then $\mathbf{B}^m \tau = 0$ for some $m \geq 0$, and the least such m equals $\text{p dim } M$.*

(b) *Suppose that $T = (\Lambda/J)_\Lambda$. Then $\text{p dim } M < \infty$ if and only if $\mathbf{B}^m \tau = 0$ for some m .*

(c) *If M is finitely generated, the integers τ_i , $1 \leq i \leq s$, can be computed as follows:*

$$\tau_i = \dim_K(A_i \otimes_\Lambda M) + \sum_{j=1}^s b_{ij} \dim_K(A_j \otimes_\Lambda M) - \sum_{l=1}^n p_{il} \dim_K(e_l M).$$

Proof. For $i = 1, \dots, s$, we have

$$\operatorname{Tor}_1^\Lambda(A_i, \Omega^1(M)) \cong \operatorname{Tor}_2^\Lambda(A_i, M) \cong \operatorname{Tor}_1^\Lambda(\Omega^1(A_i), M) \cong \bigoplus_{j=1}^s \operatorname{Tor}_1^\Lambda(A_j, M)^{b_{ij}},$$

whence $\dim_K \operatorname{Tor}_1^\Lambda(A_i, \Omega^1(M)) = \sum_{j=1}^s b_{ij} \tau_j$. This shows that $\tau(\Omega^1(M)) = \mathbf{B} \tau(M)$. It follows by induction that

$$\mathbf{B}^m \tau(M) = \tau(\Omega^m(M))$$

for all $m \geq 0$.

(a) Suppose that $\text{p dim } M = m < \infty$. Then $\Omega^m(M)$ is projective, and consequently $\operatorname{Tor}_1^\Lambda(A_i, \Omega^m(M)) = 0$ for all i . Hence, $\mathbf{B}^m \tau = \tau(\Omega^m(M)) = 0$. We must also show that if $m > 0$, then $\mathbf{B}^{m-1} \tau \neq 0$. As in the proof of Observation 2.5, $\operatorname{Tor}_m^\Lambda(T, M) \neq 0$, whence $\operatorname{Tor}_1^\Lambda(T, \Omega^{m-1}(M)) \neq 0$. Since T is a direct sum of copies of the A_i , it follows that $\operatorname{Tor}_1^\Lambda(A_i, \Omega^{m-1}(M)) \neq 0$ for some i , and therefore $\mathbf{B}^{m-1} \tau = \tau(\Omega^{m-1}(M)) \neq 0$ as desired.

(b) Suppose that $\mathbf{B}^m \tau = 0$ for some m . Then $\tau(\Omega^m(M)) = 0$, and so we have $\operatorname{Tor}_1^\Lambda(A_i, \Omega^m(M)) = 0$ for all i . Consequently, $\operatorname{Tor}_1^\Lambda(\Lambda/J, \Omega^m(M)) = 0$, which implies that $\Omega^m(M)$ is projective, and thus $\text{p dim } M \leq m$.

(c) From the resolutions $0 \rightarrow \Omega^1(A_i) \rightarrow P_i \rightarrow A_i \rightarrow 0$, we obtain long exact sequences

$$0 \rightarrow \operatorname{Tor}_1^\Lambda(A_i, M) \rightarrow \Omega^1(A_i) \otimes_\Lambda M \rightarrow P_i \otimes_\Lambda M \rightarrow A_i \otimes_\Lambda M \rightarrow 0,$$

and consequently

$$\tau_i = \dim_K(A_i \otimes_\Lambda M) + \dim_K(\Omega^1(A_i) \otimes_\Lambda M) - \dim_K(P_i \otimes_\Lambda M).$$

Since $\Omega^1(A_i) \otimes_\Lambda M \cong \bigoplus_{j=1}^s (A_j \otimes_\Lambda M)^{b_{ij}}$ and $P_i \otimes_\Lambda M \cong \bigoplus_{l=1}^n (e_l M)^{p_{il}}$, the desired formula is clear. \square

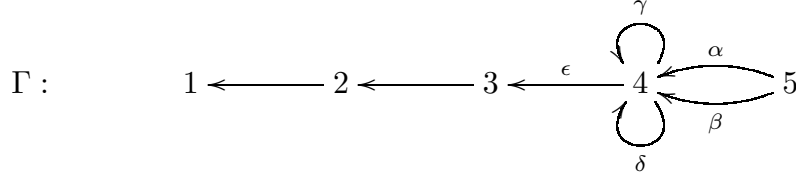
Corollary 3.2. *If d is the least nonnegative integer with the property that $\mathbf{B}^d \in M_s(\mathbb{Q}) \mathbf{B}^{d+1}$, then $\text{l Fin dim } \Lambda \leq d$.*

Proof. Observe that $\mathbf{B}^d \in M_s(\mathbb{Q}) \mathbf{B}^{d+k}$ for all $k > 0$. Hence, there exist positive integers z_k and nonnegative integer matrices $\mathbf{U}_k, \mathbf{V}_k$ such that $z_k \mathbf{B}^d = (\mathbf{U}_k - \mathbf{V}_k) \mathbf{B}^{d+k}$.

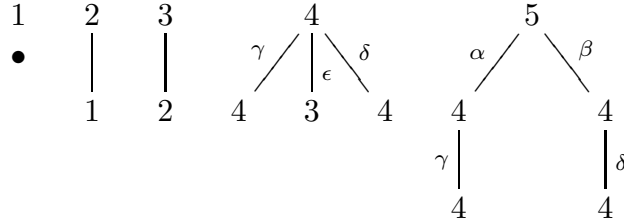
Given any left Λ -module M with $\text{p dim } M = m < \infty$, we have $\mathbf{B}^m \tau(M) = 0$ by Proposition 3.1. If $m > d$, multiply both sides of the equation $z_{m-d} \mathbf{B}^d + \mathbf{V}_{m-d} \mathbf{B}^m = \mathbf{U}_{m-d} \mathbf{B}^m$ on the right by $\tau(M)$ to obtain $z_{m-d} \mathbf{B}^d \tau(M) = 0$. But then $\mathbf{B}^d \tau(M) = 0$, contradicting Proposition 3.1. Therefore $\text{p dim } M \leq d$. \square

We conclude this section with an example illustrating the use of Proposition 3.1.

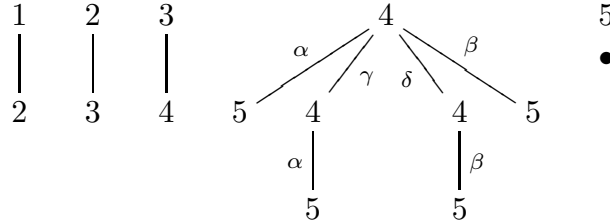
Example 3.3. Let $\Lambda = K\Gamma/I$ be the monomial relation algebra with quiver



such that the graphs of the indecomposable projective left Λ -modules are



Then the indecomposable projective right Λ -modules have graphs



Let e_1, \dots, e_5 be the primitive idempotents of Λ corresponding to the vertices of Γ , set $S_i = e_i\Lambda/e_iJ$, and let T be the right Λ -module $\Lambda/J \cong \bigoplus_{i=1}^5 S_i$. One computes that

$$\Omega^1(S_1) \cong S_2; \quad \Omega^1(S_2) \cong S_3; \quad \Omega^1(S_3) \cong S_4; \quad \Omega^1(S_5) = 0;$$

while

$$\Omega^1(S_4) \cong S_5^2 \oplus \begin{array}{c} 4 \\ \alpha \downarrow \\ 5 \end{array} \oplus \begin{array}{c} 4 \\ \beta \downarrow \\ 5 \end{array} \quad \text{and} \quad \Omega^2(S_4) \cong S_5^2 \oplus \left(\begin{array}{c} 4 \\ \alpha \downarrow \\ 5 \end{array} \right)^2 \oplus \left(\begin{array}{c} 4 \\ \beta \downarrow \\ 5 \end{array} \right)^2.$$

Thus $\text{rep}((\Lambda/J)_\Lambda) = 4$, which by Observation 2.5 yields $\text{lFin dim } \Lambda \leq 4$. (Since the left Λ -module $\Lambda/(\alpha + \beta)$ has projective dimension 4, we actually obtain equality.)

Moreover, we observe that T has finite syzygy type, the syzygy category of T being equal to $\text{add}(A_1, \dots, A_7)$ with $A_i \cong S_i$ for $i = 1, \dots, 5$ while $A_6 = \begin{array}{c} 4 \\ \alpha \downarrow \\ 5 \end{array}$ and $A_7 = \begin{array}{c} 4 \\ \beta \downarrow \\ 5 \end{array}$.

Using the notation introduced above, we thus obtain $s = 7$ and the integer 7×7 matrix

$$\mathbf{B} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

The projective covers of the A_i being P_i for $i \leq 5$ and $P_6 \cong P_7 \cong e_4\Lambda$, we finally see that $p_{ii} = 1$ for $i \leq 5$ and $p_{64} = p_{74} = 1$, while all the other exponents p_{il} are zero.

We will use Proposition 3.1 to determine the projective dimension of the left Λ -module

$$M = (\Lambda e_4 \oplus \Lambda e_5) / (\Lambda(\gamma, \alpha) + \Lambda(\beta, \delta) + \Lambda(\epsilon, 0)).$$

Clearly, $\dim_K(A_i \otimes_\Lambda M) = 0$ for $i = 1, 2, 3$ and

$$\dim_K(A_4 \otimes_\Lambda M) = \dim_K(e_4M/e_4JM) = 1 = \dim_K(A_5 \otimes_\Lambda M).$$

Moreover, we compute that

$$\dim_K(A_6 \otimes_\Lambda M) = \dim_K(e_4M/(\gamma\Lambda + \delta\Lambda + \beta\Lambda)M) = 1,$$

and analogously, $\dim_K(A_7 \otimes_\Lambda M) = 1$. In view of 3.1(c), this yields

$$\tau(M) = \tau = (0, 0, 1, 2, 0, 1, 1)^{\text{tr}}.$$

Since the last three rows of each power \mathbf{B}^m are of the form

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2^m & 2^m & 2^m \\ 0 & 0 & 0 & 0 & 2^m & 2^m & 2^m \end{pmatrix}$$

for $m \in \mathbb{N}$, we deduce that $\mathbf{B}^m \tau \neq 0$ for all m . Thus 3.1(b) tells us that $\text{p dim } M = \infty$. \square

4. A NEW CLASS OF ALGEBRAS OF FINITE GLOBAL REPETITION INDEX

As we will see, any classical order \mathcal{O} of finite global dimension d over a discrete valuation ring D has a finite dimensional satellite algebra Λ which has (left and right) global repetition index $d - 1$. This algebra, first studied by Tarsy [22] and Jategaonkar [15], and subsequently by Kirkman-Kuzmanovich [17], faithfully reflects the homological behavior of \mathcal{O} while being structurally less involved.

We start with a synopsis of problems and results pertaining to classical orders. Let D be a discrete valuation ring with uniformizing parameter π (that is, πD is the maximal ideal of D), and quotient field Q . Moreover, let \mathcal{O} be a classical order in some matrix ring

$M_n(Q)$, that is, $\mathcal{O} \subseteq M_n(Q)$ is a D -subalgebra which is finitely generated as a D -module on one hand and which generates $M_n(Q)$ over Q on the other. We will identify D with the subring $D \cdot 1$ of \mathcal{O} .

In 1970, Tarsy [21] conjectured that the global dimension of any classical order $\mathcal{O} \subseteq M_n(Q)$, whenever finite, is bounded above by $n - 1$. At least in the special case where \mathcal{O} is *tilted*, meaning that \mathcal{O} contains a complete set of n primitive orthogonal idempotents of $M_n(Q)$, this conjecture seemed to have some merit. Tarsy showed in [22] that there are only finitely many possible finite global dimensions for tilted classical orders in $M_n(Q)$, thus guaranteeing a finite upper bound on these dimensions. Moreover, in [15] Jategaonkar proved that, up to conjugation, $M_n(Q)$ contains only finitely many tilted classical orders of finite global dimension. Subsequently, Tarsy's conjecture was confirmed in the triangular case (i.e., when \mathcal{O} is conjugate to an order of the form

$$\begin{pmatrix} D & D & \cdots & D \\ (\pi^{\lambda_{2,1}}) & D & \cdots & D \\ \vdots & \ddots & \ddots & \vdots \\ (\pi^{\lambda_{n,1}}) & \cdots & (\pi^{\lambda_{n,n-1}}) & D \end{pmatrix}$$

with $\lambda_{i,j} \geq 0$) in [15], and for tilted orders in $M_n(D)$ containing the ideal $M_n(\pi D)$ in [17]. However, shortly afterwards, Fujita [5] constructed a class of tilted examples $\mathcal{O}_n \subseteq M_n(K((\pi)))$ for $n \geq 6$, where K is a field, such that $\text{gldim } \mathcal{O}_n = n$, thus refuting the conjecture even in the tilted case. Moreover, in view of an example of Rump ([20], §7, Example 3) and a class of examples exhibited by Jansen and Odenthal in [14], raising ‘ $n - 1$ ’ to ‘ n ’ (or even ‘ $n + 1$ ’) in Tarsy's conjecture will not suffice to provide a valid bound either.

The satellite algebra of \mathcal{O} which we will consider here is the factor $\Lambda = \mathcal{O}/\pi\mathcal{O}$. Clearly, Λ is a finite dimensional algebra over the residue field $K = D/\pi D$. For the case where \mathcal{O} is tilted and basic, Kirkman and Kuzmanovich assembled a list of distinguished properties of Λ , including the following [18]:

- Λ is a split basic algebra. More precisely, $\Lambda \cong K\Gamma/I$ is a binomial relation algebra, where the ideal I has a generating set consisting of paths and differences of paths. Furthermore, the quiver Γ and a generating set for I of the described form can be explicitly (and easily) computed from the valued quiver which was associated to any tilted classical order by Wiedemann and Roggenkamp [23]. (A sketch of this procedure can be found ahead of Example 4.6 below.)
- Each of the simple Λ -modules occurs as a composition factor of multiplicity 1 in each of the indecomposable projective Λ -modules. Consequently, the multiplicities of the simples are equal in all Λ -modules of finite projective dimension, which, in particular, shows that all simple modules have infinite projective dimension for $n \geq 2$.

Since \mathcal{O} is noetherian and π is a central non-zero-divisor in \mathcal{O} which is not a unit, the standard change-of-rings arguments for homological dimensions (see, e.g., [16]) apply to the pair \mathcal{O} and $\Lambda = \mathcal{O}/\pi\mathcal{O}$. They yield a helpful relationship between the little finitistic dimensions of \mathcal{O} and Λ , as was observed by Tarsy and by Green, Kirkman, and Kuzmanovich:

Proposition 4.1. $\text{l fin dim } \mathcal{O} = \text{l fin dim } \Lambda + 1$ and $\text{r fin dim } \mathcal{O} = \text{r fin dim } \Lambda + 1$.

Proof. [22], proof of Corollary 1; [8], Lemma 2.7. \square

For the big finitistic dimensions, however, these arguments only yield inequalities

$$\text{l Fin dim } \mathcal{O} \geq \text{l Fin dim } \Lambda + 1 \quad \text{and} \quad \text{r Fin dim } \mathcal{O} \geq \text{r Fin dim } \Lambda + 1.$$

In case the order \mathcal{O} has finite global dimension, or more generally, if it has finite injective dimension on both sides, we can combine Proposition 4.1 with Proposition 1.3 to obtain coincidence of the big and little finitistic dimensions of Λ , as follows.

Proposition 4.2. *If $\text{i dim } {}_{\mathcal{O}}\mathcal{O}$ and $\text{i dim } \mathcal{O}_{\mathcal{O}}$ are both finite, then*

$$\begin{aligned} \text{l Fin dim } \Lambda &= \text{r Fin dim } \Lambda = \text{l fin dim } \Lambda = \text{r fin dim } \Lambda \\ &= \text{i dim } {}_{\Lambda}\Lambda = \text{i dim } \Lambda_{\Lambda} = \text{p dim } E({}_{\Lambda}(\Lambda/J)) = \text{p dim } E((\Lambda/J)_{\Lambda}) \\ &= \text{l fin dim } \mathcal{O} - 1 = \text{r fin dim } \mathcal{O} - 1. \end{aligned}$$

In particular, if $\text{gl dim } \mathcal{O} = d < \infty$, then $d \geq 1$ and all the above dimensions equal $d - 1$.

Proof. Since the order \mathcal{O} is not divisible by π on either side, it is not injective as a right or left \mathcal{O} -module. Therefore, π being a central non-zero-divisor in \mathcal{O} , it follows from [16], Theorem 205 that

$$\text{i dim } {}_{\Lambda}\Lambda \leq \text{i dim } {}_{\mathcal{O}}\mathcal{O} - 1 < \infty,$$

and likewise that $\text{i dim } \Lambda_{\Lambda} < \infty$. Thus, by Proposition 1.3,

$$\begin{aligned} \text{l Fin dim } \Lambda &= \text{r Fin dim } \Lambda = \text{l fin dim } \Lambda = \text{r fin dim } \Lambda \\ &= \text{i dim } {}_{\Lambda}\Lambda = \text{i dim } \Lambda_{\Lambda} = \text{p dim } E({}_{\Lambda}(\Lambda/J)) = \text{p dim } E((\Lambda/J)_{\Lambda}). \end{aligned}$$

If d' is the common value of these dimensions, Proposition 4.1 shows that $\text{l fin dim } \mathcal{O} = \text{r fin dim } \mathcal{O} = d' + 1$.

The final statement of the proposition is clear. \square

We shall need the following well known variant of Schanuel's Lemma, which holds for modules over an arbitrary artinian ring: If $0 \rightarrow L \rightarrow P \rightarrow N \rightarrow 0$ is a short exact sequence of Λ -modules with P projective, then $L \cong \Omega^1(N) \oplus Q$ for some projective Λ -module Q . This conclusion can be obtained from Schanuel's Lemma together with the Krull-Schmidt Theorem, or from [1], Lemma 17.17.

Theorem 4.3. *If M is a nonzero left Λ -module with $\text{p dim } {}_{\mathcal{O}}M = m < \infty$, then $m > 0$ and $\Omega_{\Lambda}^{m-1}(M) \cong \Omega_{\Lambda}^{m+1}(M) \oplus Q$ for some projective module Q . In particular, $\text{rep}({}_{\Lambda}M) \leq m - 1$, and if M is finitely generated, then ${}_{\Lambda}M$ has finite syzygy type.*

Proof. Since M is torsion as an \mathcal{O} -module, $m \geq 1$.

Suppose first that $m = 1$, and write $M \cong P/K$ for some projective \mathcal{O} -modules $K \subseteq P$. Since $\pi M = 0$, there is an exact sequence

$$0 \rightarrow K/\pi P \rightarrow P/\pi P \rightarrow M \rightarrow 0$$

of Λ -modules with $P/\pi P$ projective, and hence $K/\pi P \cong \Omega_\Lambda^1(M) \oplus Q_1$ for some projective Λ -module Q_1 . There is also an exact sequence

$$0 \rightarrow \pi P/\pi K \rightarrow K/\pi K \rightarrow K/\pi P \rightarrow 0$$

of Λ -modules with $K/\pi K$ projective, and so $\pi P/\pi K \cong \Omega_\Lambda^1(K/\pi P) \oplus Q$ for some projective Λ -module Q . Consequently, $\pi P/\pi K \cong \Omega_\Lambda^2(M) \oplus Q$. Since π is a non-zero-divisor on P , we have $\pi P/\pi K \cong P/K \cong M$ as well.

Now assume that $m > 1$, and consider the exact sequence $0 \rightarrow L \rightarrow Q_0 \rightarrow M \rightarrow 0$ of Λ -modules where Q_0 is a Λ -projective cover of M and $L = \Omega_\Lambda^1(M)$. Observe that $\text{p dim } \mathcal{O} = 1$ and so $\text{p dim } \mathcal{O} Q_0 \leq 1$. Since $m \geq 2$, it follows from the long exact sequence for $\text{Ext}_{\mathcal{O}}$ that $\text{p dim } \mathcal{O} L = m - 1$. By induction, $\Omega_\Lambda^{m-2}(L) \cong \Omega_\Lambda^m(L) \oplus Q$ for some projective module Q , and therefore $\Omega_\Lambda^{m-1}(M) \cong \Omega_\Lambda^{m+1}(M) \oplus Q$.

The final conclusions of the theorem now follow easily. \square

Corollary 4.4. *If $\text{gl dim } \mathcal{O} = d < \infty$, then $\text{lgl rep } \Lambda = \text{rgl rep } \Lambda = d - 1$ and all finitely generated Λ -modules have finite syzygy type.*

Proof. Suppose that $\text{gl dim } \mathcal{O} = d < \infty$. Theorem 4.3 then shows that $\text{lgl rep } \Lambda$ and $\text{rgl rep } \Lambda$ are bounded above by $d - 1$, and that all finitely generated Λ -modules have finite syzygy type. To obtain the missing inequalities, we use Proposition 4.1 and Observation 2.5 to obtain

$$d - 1 = \text{l fin dim } \Lambda \leq \text{rep}((\Lambda/J)_\Lambda) \leq \text{rgl rep } \Lambda,$$

and similarly $d - 1 \leq \text{lgl rep } \Lambda$. \square

The next consequence of Theorem 4.3 addresses the Poincaré-Betti series of a pair (M, N) of Λ -modules, i.e. the formal power series

$$\sum_{i=0}^{\infty} (-1)^i \dim_K \text{Ext}_\Lambda^i(M, N) \cdot t^i.$$

Corollary 4.5. *If \mathcal{O} has finite global dimension, then the Poincaré-Betti series of any pair of Λ -modules is a rational function.*

Proof. Combine Corollary 4.4 with Wilson's main result in [24]. \square

We now briefly sketch the road that leads from a basic tiled classical order \mathcal{O} to a presentation of $\Lambda = \mathcal{O}/\pi\mathcal{O}$ in terms of quiver and relations, as it was communicated to us by Kirkman and Kuzmanovich [18]. We would like to warn the reader, however, that the quivers we use are duals of those considered in [17] and [23].

After a conjugation, we may assume that $\mathcal{O} \subseteq M_n(D)$ and that the diagonal matrix units e_{ii} , $1 \leq i \leq n$, all lie in \mathcal{O} . Now consider the ordinary quiver Γ of the (semiperfect) ring \mathcal{O} : It consists of n vertices $1, \dots, n$ corresponding to the n non-isomorphic indecomposable projective left \mathcal{O} -modules $P_i = \mathcal{O}e_{ii}$, and there is an arrow $i \rightarrow j$ provided that P_j is isomorphic to a direct summand of the projective cover of $\text{rad } P_i$. (This is in accordance

with the usual convention, since the multiplicity of P_j as a direct summand of the projective cover of $\text{rad } P_i$ is at most 1 in the present situation.) We attach to any arrow $i \xrightarrow{\alpha} j$ the value $v(\alpha) = k$ where k is the least nonnegative integer such that $\pi^k P_j \subseteq \text{rad } P_i$ (so that $D\pi^k e_{ji}$ corresponds, via right multiplication, to $\text{Hom}_{\mathcal{O}}(P_j, \text{rad } P_i)$). Thus $D\pi^k e_{ji} = e_{jj}\mathcal{O}e_{ii}$ if $i \neq j$, whereas $k = 1$ if $i = j$. This process yields the ‘valued quiver’ $\tilde{\Gamma}$ of \mathcal{O} . (The order \mathcal{O} can actually be retrieved from $\tilde{\Gamma}$ – see [23].) The value function v is extended additively from arrows to paths. Thus the value of a path $q = \alpha_m \alpha_{m-1} \cdots \alpha_1$ in $\tilde{\Gamma}$, where the α_i are arrows, is $v(q) = \sum_{i=1}^m v(\alpha_i)$.

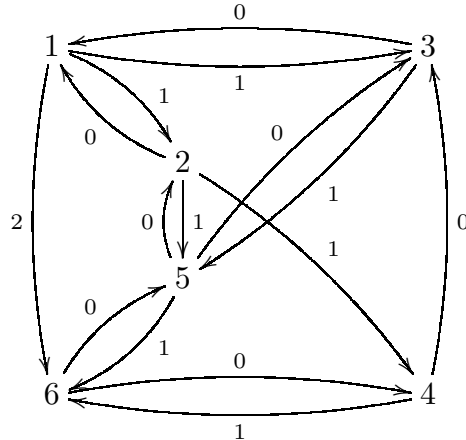
We now specialize to the situation where $\tilde{\Gamma}$ has no loops. In that case the quiver of $\Lambda = \mathcal{O}/\pi\mathcal{O}$ coincides with the quiver Γ as above, and the following set G of paths and differences of paths in $K\Gamma$ generates an ideal $I \subseteq K\Gamma$ such that $\Lambda \cong K\Gamma/I$. A path q , with starting point i and endpoint j say, belongs to G if and only if there exists a path q' from i to j in $\tilde{\Gamma}$ such that $v(q') < v(q)$. Moreover, given two paths p and q with coinciding starting points and coinciding endpoints, the difference $p - q$ belongs to G if and only if $v(p) = v(q)$.

Since, given a tiled classical order \mathcal{O} , the algebra $\Lambda = \mathcal{O}/\pi\mathcal{O}$ is substantially easier to handle, this latter algebra serves as an excellent tool for the investigation of \mathcal{O} . In our first illustration of this principle, we exhibit a classical order with differing left and right finitistic dimensions.

Example 4.6. Let D be any discrete valuation ring with uniformizing parameter π and quotient field Q , and consider the following tiled classical order $\mathcal{O} \subseteq M_6(Q)$:

$$\mathcal{O} = \begin{pmatrix} D & D & D & D & D & D \\ (\pi) & D & (\pi) & (\pi) & D & D \\ (\pi) & (\pi) & D & D & D & D \\ (\pi^2) & (\pi) & (\pi^2) & D & (\pi) & D \\ (\pi^2) & (\pi) & (\pi) & (\pi) & D & D \\ (\pi^2) & (\pi^2) & (\pi^2) & (\pi) & (\pi) & D \end{pmatrix}$$

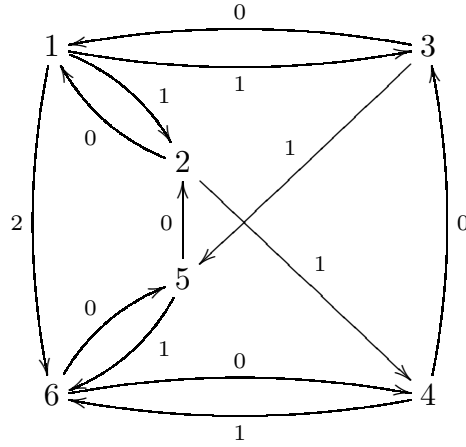
The valued quiver of \mathcal{O} is



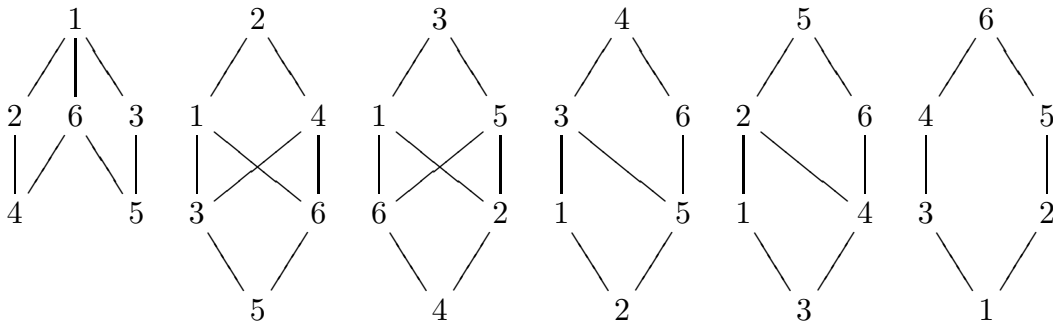
and $\Lambda = \mathcal{O}/\pi\mathcal{O}$ has the same quiver (non-valued) and relations p and $p - q$, for suitable paths p and q such that the indecomposable projective right Λ -modules are as in Example 2.8. Combining our computations in the latter example with Proposition 4.1, we thus obtain $\text{l fin dim } \mathcal{O} = 4$ and $\text{r fin dim } \mathcal{O} = 1$. \square

In our final example, we construct a tiled classical order \mathcal{O} of infinite global dimension such that $\Lambda = \mathcal{O}/\pi\mathcal{O}$ is Gorenstein, i.e., such that $\text{i dim } \Lambda = \text{i dim } \Lambda_\Lambda < \infty$. The fact that $\text{gl dim } \mathcal{O} = \infty$ will be established by pinpointing a left Λ -module which does not satisfy the conclusion of Theorem 4.3.

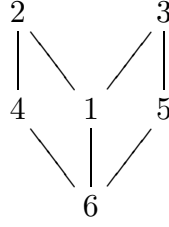
Example 4.7. Again, let D be any discrete valuation ring with uniformizing parameter π and quotient field Q , and let $\mathcal{O} \subseteq M_6(Q)$ be the tiled classical order which has valued quiver



Then the quiver Γ of $\Lambda = \mathcal{O}/\pi\mathcal{O}$ is the non-valued version of the one shown and Λ is of the form $K\Gamma/I$, where I is generated by paths and differences of paths, such that the indecomposable projective left Λ -modules have the following graphs:

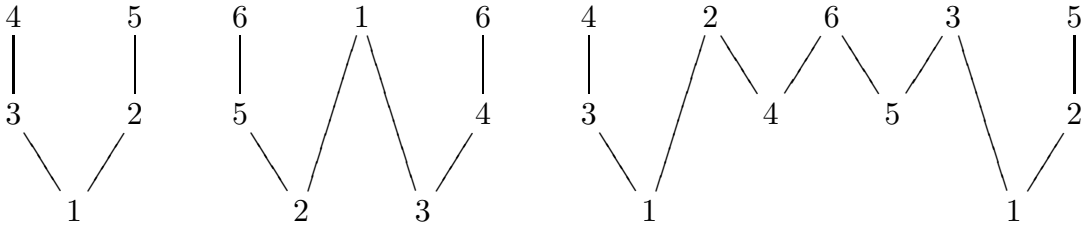


In particular, the only indecomposable injective object in $\Lambda\text{-mod}$ which fails to be projective is the injective envelope E_6 of $\Lambda e_6/J e_6$, namely the unique left Λ -module with graph



One checks that $\Omega^1(E_6) \cong \Lambda e_1$, whence $\text{p dim } E_6 = 1$. Thus the minimal injective cogenerator for $\Lambda\text{-mod}$ has projective dimension 1. Analogously, one computes the projective dimension of the minimal injective right cogenerator to be 1, which yields $\text{i dim } {}_\Lambda \Lambda = \text{i dim } \Lambda_\Lambda = 1$ by duality. Furthermore, $\text{l fin dim } \Lambda = \text{r fin dim } \Lambda = 1$ by Proposition 1.3. Consequently, Proposition 4.1 tells us that $\text{l fin dim } \mathcal{O} = \text{r fin dim } \mathcal{O} = 2$.

In order to check that $\text{gldim } \mathcal{O} = \infty$, it thus suffices to procure a left Λ -module M which violates the conclusion of Theorem 4.3 for $m = 2$. One can readily verify that each of the simple left Λ -modules qualifies for that purpose. We pick one with a ‘slim’ projective resolution, namely $M = \Lambda e_6 / J e_6$. Then the first three syzygies $\Omega^1(M)$, $\Omega^2(M)$, $\Omega^3(M)$ have the following graphs, respectively:



Clearly, these syzygies are indecomposable and $\Omega^1(M) \not\cong \Omega^3(M)$, which shows that indeed $\text{p dim}_{\mathcal{O}} M > 2$. (The repetition index of M as a Λ -module is, in fact, infinite.) \square

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